

General Uniformity of Zeta Functions

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— In memory of Prof. Dr. F. Hirzebruch

Abstract. Using analytic torsion associated to stable bundles, we introduce zeta functions for compact Riemann surfaces. To justify the well-definedness, we analyze the degenerations of analytic torsions at the boundaries of the moduli spaces, the singularities of analytic torsions at Brill-Noether loci, and the asymptotic behaviors of analytic torsions with respect to the degree. These new yet intrinsic zetas, both abelian and non-abelian, are expected to play key roles to understand global analysis and geometry of Riemann surfaces, such as the Tamagawa number conjecture for Riemann surfaces, searched by Atiyah-Bott, and the volumes formula of moduli spaces of Witten. Relating to this, in our theory on special uniformity of zetas, we will first construct a symmetric zetas based on abelian zetas and group symmetries, then conjecture that our non-abelian zetas coincide with these later zetas with symmetries. All this, together with that for zetas of number fields and function fields, then consists of our theory of general uniformity of zetas.

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1 Zeta Functions for Riemann surfaces

1.1 Regularized Integrations

Let M be a compact Riemann surface of genus $g > 1$ with fundamental group $\pi_1(M)$. Fixed a normalized smooth volume form ω on M so that $\int_M \omega = 1$. For a line bundle L of degree d , by definition, an ω -admissible metric h on L is a hermitian metric on L such that $c_1(L, h) = d \cdot \omega$. One checks that ω -admissible metrics always exist and for a fixed line bundle they are parametrized by positive reals. For our purpose, fix a point $P_0 \in M$ and use the normalized Green function ($[L]$) to define the ω -admissible metric h_A on the line bundle $A = A_M$ corresponding to the invertible sheaf $\mathcal{O}_M(P_0)$. (From now on, we will not make distinctions between bundles and locally free sheaves.) Moreover, fix a conformal metric $\tau = \tau_M$ on M such that its induced metric on the canonical line bundle K_M is ω -admissible and whole volume is given by $2\pi(2g - 2)$.

Let $\mathcal{M}_{M,r}(d)$ be the moduli space of stable bundles of rank r and degree d on M . Then we have natural isomorphisms, for all $m \in \mathbb{Z}$,

$$\mathcal{M}_{M,r}(0) \simeq \mathcal{M}_{M,r}(mr), \quad V \mapsto V \otimes A^{\otimes m}.$$

By a result of Narasimhan-Seshadri [NS], points V of $\mathcal{M}_{M,r}(0)$ are in one-to-one correspondence to irreducible unitary representations ρ of $\pi_1(M)$. (Here for simplicity, we assume that $g \geq 2$.) This will be indicated as $V = V_\rho$.

Use the uniformization, denote the induced hermitian metric by h_ρ on V_ρ from the standard one on \mathbb{C}^r . As such, then for any $V \in \mathcal{M}_{M,r}(rm)$, we have the associated canonical metric $h_V = h_\rho \otimes h_A^{\otimes m}$ on $V = V_\rho \otimes A^{\otimes m}$ induced from the canonical metric h_ρ on V_ρ and the canonical metric h_A on A constructed above. For simplicity, denote such a metrized bundle (resp. Riemann surface M) by \overline{V} (resp. \overline{M}).

Denote by $\tau(M, V)$ the Ray-Singer analytic torsion associated to the hermitian vector bundle \overline{V} over \overline{M} . Then the determinant line bundle λ on $\mathcal{M}_{M,r}(rm)$, whose fiber at $V \in \mathcal{M}_{M,r}(rm)$ is given by $\det H^0(M, V) \otimes \det H^1(M, V)^{\otimes -1}$, together with the associated Quillen metric $h_Q(V, M)$, induced from that of $\overline{V}/\overline{M}$, is a definite metrized polarization on $\mathcal{M}_{M,r}(rm)$. As above, denote the resulting metrized line bundle by $\overline{\lambda}$.

Let \mathcal{V}_m be the universal Poincare vector bundle on $\mathcal{M}_{M,r}(rm)$. Then with respect to the projection $q : M \times \mathcal{M}_{M,r}(rm) \rightarrow \mathcal{M}_{M,r}(rm)$, assume that, from the short exact sequence of coherent sheaves

$$0 \rightarrow \mathcal{V}_m \rightarrow \mathcal{V}_m \otimes A^{\otimes d} \rightarrow \mathcal{V}_m \otimes A^{\otimes d} / \mathcal{V}_m \rightarrow 0,$$

we obtain the exact sequence of **vector bundles** for higher direct images

$$0 \rightarrow q_* \mathcal{V}_m \rightarrow q_* (\mathcal{V}_m \otimes A^{\otimes d}) \xrightarrow{\gamma} q_* (\mathcal{V}_m \otimes A^{\otimes d} / \mathcal{V}_m) \rightarrow R^1 q_* \mathcal{V}_m \rightarrow 0.$$

Denote by $W_{M,r}^{\geq i}(d)$ the determinantal variety associated to γ . Then one checks that the support of $W_{M,r}^{\geq i}(d)$ coincides with the so-called Brill-Noether locus consisting of these whose h^0 are at least i . (See e.g., [ACGH, p.176].) Indeed, $W_{M,r}^{\geq i}(d)$ are normal subvarieties of $\mathcal{M}_{M,r}(d)$ and $T(V, M)$ is a smooth function on

$$W_{M,r}^i(d) = W_{M,r}^{\geq i}(d) \setminus W_{M,r}^{\geq (i+1)}(d)$$

when $d \in r\mathbb{Z}$. By an abuse of notation, denote by $d\mu$ the volume forms on $W_{M,r}^{\geq i}(d)$ induced from the polarization λ .

With all this, we are ready to introduce the regularized integration by

$$\begin{aligned} & \int_{\mathcal{M}_{M,r}(rm)}^{\#} \left(e^{\tau(M,V)} - 1 \right) (e^{-s})^{\chi(M,V)} d\mu \\ &:= \sum_{i=0}^{\infty} \int_{W_{M,r}^i(rm)} \left(e^{\tau(M,V)} \right) (e^{-s})^{\chi(M,V)} d\mu - \int_{\mathcal{M}_{M,r}(rm)} (e^{-s})^{\chi(M,V)} d\mu. \end{aligned}$$

1.2 Zetas for Riemann Surfaces

Let M be a compact Riemann surface and fix $r \in \mathbb{N}$. We define the *rank r zeta function for M* by

$$\widehat{\zeta}_{M,r}(s) := \sum_{m=-\infty}^{\infty} \int_{\mathcal{M}_{M,r}(rm)}^{\#} \left(e^{\tau(M,V)} - 1 \right) (e^{-s})^{\chi(M,V)} d\mu, \quad \operatorname{Re}(s) > 0.$$

The main result of this paper is the following

Theorem 1. (Zeta Facts) (i) $\widehat{\zeta}_{M,r}(s)$ is well defined and admits a (unique) meromorphic continuation to the whole complex s -plane.

(ii) **(Functional Equation)**

$$\widehat{\zeta}_{M,r}(-s) = \widehat{\zeta}_{M,r}(s).$$

(iii) **(Singularities)** $\widehat{\zeta}_{M,r}(s)$ admits only one singularity, namely, a simple pole at $s = 0$ with the residue

$$\text{Res}_{s=0} \widehat{\zeta}_{M,r}(s) = \text{Vol}(\mathcal{M}_{M,r}(0))$$

where $\text{Vol}(\mathcal{M}_{M,r}(0))$ denotes the volume of the moduli space $\mathcal{M}_{M,r}(0)$ with respect to the volume $d\mu$.

1.3 Zeta Facts I: Formal Aspect

Before justifying the convergence of our regularized integrations appeared in the definition of new zeta functions, let us formally establish the functional equation and find out the singularities of these zetas and hence calculate the associated residues when applicable.

For this purpose, we first recall some basic facts about analytic torsions. Let $\overline{V}/\overline{M}$ be a metrized vector bundle over a metrized Riemann surface. Denote the associated Laplacian by D_V on the associated space of L^2 sections $L^2(\overline{M}, \overline{V})$ of V on M . From the Fredholm theory, the spectrum of D_V is a purely discrete sequence

$$0 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots, \quad \lambda_n \sim \frac{1}{r(g-1)}n$$

with corresponding eigenfunctions $\{e_n(z, V)\}$ forming a complete orthonormal basis. Accordingly, we can define for $\text{Re}(\lambda) > 0$ and $\text{Re}(s) > 1$ the spectrum zeta function

$$\zeta_\lambda(s, V) := \text{Tr}(D_L + \lambda)^{-s} = \sum_{n=1}^{\infty} \frac{1}{(\lambda + \lambda_n)^s}$$

and more generally for any $c \geq 0$,

$$\zeta_\lambda^c(s, V) := \sum_{\lambda > c} \frac{1}{(\lambda + \lambda_n)^s}.$$

We have the following

Theorem 2. ([RS]) (i) For fixed $c \geq 0$, $\zeta_\lambda^c(s, V)$ has an analytic continuation to the half plane $\operatorname{Re}(\lambda) \geq c$ and a meromorphic continuation to the whole s -plane with only a simple pole at $s = 1$ with residue $r(g - 1)$.
(ii) For $\operatorname{Re}(\lambda) > \lambda_*$ the smallest non-zero eigenvalue of D_V ,

$$\zeta_\lambda^0(s, V) = \zeta_\lambda(s, V) - h^0(M, V)\lambda^{-s}$$

has an analytic continuation through the s -plane with

$$\zeta_\lambda^0(0, V) + h^0(M, V) = -\left(\lambda + \frac{1}{3}\right)n(g - 1) + \frac{1}{2}d(V).$$

(iii) **(Duality)** Equipped the dual bundle V^\vee with the dual metric, then

$$\zeta_\lambda^c(s, V) = \zeta_\lambda^c(s, K_M \otimes V^\vee)$$

Based on this, following Ray-Singer ([RS]), define the *analytic torsion* for $\overline{V}(\overline{M})$ by

$$T(V) := T(\overline{M}; \overline{V}) := e^{\tau(V)}$$

with

$$\tau(V) := \tau(M, V) := \tau(\overline{M}; \overline{V}) := \frac{d}{ds} \zeta_\lambda^0(s, V)|_{s=\lambda=0}.$$

It is well known then that $T(V)$ may be viewed as a regularized determinant of the Laplacian D_V . That is to say, formally, we have

$$T(V) = \det'(D_V) = \prod'_{n: \lambda_n > 0} \lambda_n.$$

Proposition 3. (Functional Equation) Assuming the convergence,

$$\widehat{\zeta}_{M,r}(-s) = \widehat{\zeta}_{M,r}(s).$$

To establish it, motivated by our works for number fields ([W]), let us divide the summation in the definition of our zetas into the following groups:

$$\begin{aligned} \widehat{\zeta}_{M,r}(s) &:= \sum_{m \in \mathbb{Z}} \int_{\mathcal{M}_{M,r}(rm)}^\# \left(e^{\tau(M,V)} - 1 \right) (e^{-s})^{\chi(M,V)} d\mu \\ &= I(s) + II(s) + III(s) - IV(s) \end{aligned}$$

with

$$\begin{aligned} I(s) &= \sum_{m=0}^{2g-2} \int_{\mathcal{M}_{M,r}(rm)}^\# e^{\tau(M,V)} (e^{-s})^{\chi(M,V)} d\mu, \\ II(s) &= \sum_{m < 0} \int_{\mathcal{M}_{M,r}(rm)} \left(e^{\tau(M,V)} \right) (e^{-s})^{\chi(M,V)} d\mu, \\ III(s) &= \sum_{m > 2g-2} \int_{\mathcal{M}_{M,r}(rm)} \left(e^{\tau(M,V)} - 1 \right) (e^{-s})^{\chi(M,V)} d\mu, \\ IV(s) &= \sum_{m \leq 2g-2} \int_{\mathcal{M}_{M,r}(rm)} (e^{-s})^{\chi(M,V)} d\mu \end{aligned}$$

First for I , we divide it further as follows:

$$\begin{aligned}
I(s) &= \sum_{m=0}^{2g-2} \int_{\mathcal{M}_{M,r}(rm)}^{\#} e^{\tau(M,V)} (e^{-s})^{\chi(M,V)} d\mu \\
&= \left(\sum_{m=0}^{(g-1)-1} + \sum_{m=g-1} + \sum_{m=g}^{2g-2} \right) \int_{\mathcal{M}_{M,r}(rm)}^{\#} e^{\tau(M,V)} (e^{-s})^{\chi(M,V)} d\mu \\
&= \sum_{m=0}^{(g-1)-1} \int_{\mathcal{M}_{M,r}(rm)}^{\#} e^{\tau(M,V)} (e^{-s})^{\chi(M,V)} d\mu \\
&\quad + \sum_{m=(g-1)-1}^0 \int_{V \in \mathcal{M}_{M,r}(rm)}^{\#} e^{\tau(M, K_M \otimes V^\vee)} (e^{-s})^{\chi(M, K_M \otimes V^\vee)} d\mu \\
&\quad + \sum_{m=g-1} \int_{\mathcal{M}_{M,r}(rm)}^{\#} e^{\tau(M,V)} (e^{-s})^0 d\mu \\
&= \sum_{m=0}^{(g-1)-1} \int_{\mathcal{M}_{M,r}(rm)}^{\#} e^{\tau(M,V)} \left((e^{-s})^{\chi(M,V)} + (e^s)^{\chi(M,V)} \right) d\mu \\
&\quad + \sum_{m=g-1} \int_{\mathcal{M}_{M,r}(rm)}^{\#} e^{\tau(M,V)} d\mu
\end{aligned}$$

by duality, i.e., Thm 2.iii) for analytic torsions and the ordinary duality on cohomology groups. Thus clearly, $I(-s) = I(s)$.

Next, for $II(s)$, using the duality, we have

$$\begin{aligned}
II(s) &= \sum_{m < 0} \int_{V \in \mathcal{M}_{M,r}(rm)} \left(e^{\tau(M, K_M \otimes V^\vee)} \right) (e^{-s})^{-\chi(X, K_M \otimes V^\vee)} d\mu \\
&= \sum_{m > 2g-2} \int_{\mathcal{M}_{M,r}(rm)} \left(e^{\tau(M,V)} \right) (e^s)^{\chi(M,V)} d\mu \\
&= III(-s) + V(s)
\end{aligned}$$

where

$$V(s) = \sum_{m > 2g-2} \int_{\mathcal{M}_{M,r}(rm)} (e^s)^{\chi(M,V)} d\mu.$$

Finally, let us calculate $IV(s)$ and $V(s)$. By definition and the Riemann-Roch, we have

$$\begin{aligned}
IV(s) &= \sum_{m \leq 2g-2} \int_{\mathcal{M}_{M,r}(rm)} (e^{-s})^{\chi(M,V)} d\mu = \sum_{m \leq 2g-2} \int_{\mathcal{M}_{M,r}(rm)} (e^{-s})^{r[m-(g-1)]} d\mu \\
&= \sum_{m \leq 2g-2} (e^{-s})^{r[m-(g-1)]} \cdot \text{Vol}(\mathcal{M}_{M,r}(0)) \\
&= \frac{(e^{rs})^{(1-g)}}{1 - e^{rs}} \cdot \text{Vol}(\mathcal{M}_{M,r}(0)) = \frac{(e^{-rs})^g}{e^{-rs} - 1} \cdot \text{Vol}(\mathcal{M}_{M,r}(0))
\end{aligned}$$

and similarly

$$\begin{aligned}
V(s) &= \sum_{m > 2g-2} \int_{\mathcal{M}_{M,r}(rm)} (e^s)^{\chi(M,V)} d\mu \\
&= \sum_{m > 2g-2} (e^s)^{r[m-(g-1)]} \cdot \text{Vol}(\mathcal{M}_{M,r}(0)) \\
&= \frac{(e^{rs})^g}{1 - e^{rs}} \cdot \text{Vol}(\mathcal{M}_{M,r}(0))
\end{aligned}$$

Clearly,

$$V(-s) = -VI(s).$$

This then formally completes the proof of the functional equation. Consequently, if $III(s)$ admits an analytic continuation to the whole s -plane, a fact which will be established in the next subsection, we then conclude the following

Proposition 4. (Singularity and Residue) *Assume the convergence. Then $\hat{\zeta}_{M,r}(s)$ admits only a single singularity, namely, a simple pole at $s = 0$ with residue*

$$\text{Res}_{s=0} \hat{\zeta}_{M,r}(s) = \text{Vol}(\mathcal{M}_{M,r}(0)).$$

Remarks. This definition of our zetas for Riemann surfaces are motivated by our works on zetas for numbers and curves over finite fields ([W]). To understand our current definition, we mention the follows:

- (i) Instead of working on $\mathcal{M}_{M,r}(d)$ for all d , we take only these which are multiples of the rank r , namely, $d \in r\mathbb{Z}$. This is due to the fact that over finite fields, such and only such a pure selection would guarantee the Riemann Hypothesis.
- (ii) The summation on all $d \in r\mathbb{Z}$, instead of $d \in r\mathbb{Z}_{\geq 0}$, is motivated by our zetas for number field: after all analytic torsions do not satisfy the vanishing but satisfy the duality.

1.4 Zeta Facts II: Analytic Aspect

To establish the zeta facts for our zeta functions, two types of convergences should be justified properly. Namely, the one for regularized integrations over the moduli spaces $\mathcal{M}_{M,r}(rm)$ for a fixed m , and the other for the infinite sum on $m, m > 2g - 2$ appeared in $III(s)$. As we will see below, these two are very different in nature: Technically, for the first type, we need to see how the analytic torsions $T(V)$ degenerate when h^0 jump; while for the second, we need to understand how the analytic torsions $T(V \otimes A^{\otimes m})$ behave when $m \rightarrow \infty$.

1.4.1 Degenerations of Analytic Torsions

In this subsection, we will establish the convergence of the regularized integration

$$\int_{\mathcal{M}_{M,r}(rm)}^{\#} \left(e^{\tau(M,V)} - 1 \right) (e^{-s})^{\chi(M,V)} d\mu$$

for each fixed m . There are a few issues here: First, $\mathcal{M}_{M,r}(rm)$ are not compact; second, $W_{M,r}^{\geq i}(rm)$ are not smooth; and finally, $T(V)$ are not smooth on $W_{M,r}^{\geq i}(rm)$.

A. Non-Compactness This is not really that serious, thanks to the classical works done. In fact, using Mumford's GIT, we now have a natural compactification $\overline{\mathcal{M}_{M,r}(rm)}$ in terms of Seshadri's equivalences of semi-stable bundles. Denote by $\partial(\mathcal{M}_{M,r}(rm)) := \overline{\mathcal{M}_{M,r}(rm)} \setminus \mathcal{M}_{M,r}(rm)$ the associated boundary. It is well-known that this boundary is of much higher co-dimension, and hence does not cause any serious trouble for the integration.

B. Geometric Singularities

Note that for any open subset $U \subset \mathcal{M}_{M,r}(rm)$, if h^0 is a constant on U , then $T(V)$ is smooth on U . Thus we only need to consider the case when $0 \leq m \leq g-1$ with the duality for analytic torsions in mind.

From the structure of determinantal varieties, it is well-known that the singularities of $W_{M,r}^{\geq i}(rm)$ is contained in $W_{M,r}^{\geq (i+1)}(rm)$, which has codimension at least 2 except in the case when $m = g-1$. Thus, in our discussion, we will use the most complicated level, i.e., $m = g-1$, to show how the convergence can be established. For other levels, which are much simpler, a consideration using the following insertion formula for analytic torsions ([AGBMNV]) is sufficient to complete the argument. (For unknown notations, please consult [F].)

Theorem 5. (see e.g. [F, Thm 4.13]) *Let L be a line bundle of degree $d \geq g$ with an admissible metric h . Then for all stable bundles $V = V_\rho$ with uniformizing metric, $h^1(M, V_\rho \otimes L) = 0$ and for any points $x_1, \dots, x_{N:=d+1-g} \in M$:*

$$T(V_\rho \otimes L) = \varepsilon_d(M) \frac{T(V_\rho \otimes L(-\sum_{i=1}^N P_i))}{\det(B(x_i, \bar{x}_j; V_\rho \otimes L))} \cdot \frac{\prod_{i < j}^N P(x_i, x_j)^{2r}}{\prod_{i=1}^m h(x_i)^r}$$

where $\varepsilon_d(\rho)$ is a constant depending only on d and M .

C. Analytic Singularities

So from now on, we concentrate on the level $g-1$. For this, we recall some facts on both abelian and non-abelian theta functions. The explosion

here follows closely that of Fay [F] (please check the meaning of the unknown notations below in [F] as well).

Theorem 6. (Theta Functions [F, Thm 1.6]) *Let L be a fixed line bundle with $h^0(\chi(s) \otimes L)$ constant for s in some neighborhood V containing $0 \in \mathbb{C}^N$; choose $\{\omega_i\}, \{\omega_i^*\}$ bases for $H^0(\chi(s) \otimes L), H^0(\chi^*(s) \otimes KL^{-1})$ with $\{M^{-1}(z, s)\omega_i\}, \{^t M(z, s)\omega_i^*\}$ holomorphic in $s \in V$. Then*

(i) for $s \in V$,

$$T(\chi \otimes L) = U(\chi(s)) |f(s)|^2 \det(\langle \omega_i, \omega_j \rangle_{\chi(s) \otimes L}) \det(\langle \omega_i^*, \omega_j^* \rangle_{\chi^*(s) \otimes KL^{-1}})$$

where $f(s)$ is a holomorphic function on V depending on L , the fixed potential U , the bases $\{\omega_i\}, \{\omega_j^\}$, and the metrics $h, I \otimes h$ and ρ . In particular,*

(ii) in a neighborhood of any point $\chi(0)$ where $h^0(\chi(s) \otimes \Delta) = 0$ with Δ a Riemann divisor class satisfying $h^0(M, \Delta) = 0$,

$$T(\chi(s) \otimes L) = c_0^l(h, \rho) U(\chi(s)) |\theta(\chi(s))|^2$$

with $\theta(\chi(s)) = \theta(s)$ holomorphic in s and independent of the metrics h, ρ .

Theorem 7. (Vanishing of Non-Abelian Theta, [F, Prop 4.7, Thm 4.8]) *There exists a holomorphic section θ_r of the determinant line bundle λ on $\mathcal{M}_{M,r}(0)$ such that*

- (i) $\theta_r(V_\rho) = 0$ if and only if $h^0(M, \text{End} V_\rho \otimes \Delta) > 0$;*
- (ii) $\frac{\theta_r(E_\rho) \theta(\det E_\rho)^2}{\theta(\rho)^{2r}}$ is a meromorphic function on $\mathcal{M}_{M,r}(0)$;*
- (iii) As a bundle on $\mathcal{M}_{M,r}(0)$, $\lambda \simeq K_{\mathcal{M}_{M,r}(0)}^\vee$, the dual of the canonical line bundle of $\mathcal{M}_{M,r}(0)$.*
- (iv) The section θ_r vanishes to order n at any representation $V_\rho \in \mathcal{M}_{M,r}(0)$ with $h^0(V_\rho \otimes \Delta) = n$. The tangent cone to (θ_r) at V_ρ is the sub variety of $\overline{w} = \sum_{i=1}^{\dim \mathcal{M}_{M,r}(0)} s_i \overline{w}_i(z, \text{End} V_\rho) \in \overline{H^0(M, K_M \otimes \text{End} V_\rho)}$ given by*

$$\det_{1 \leq i, j \leq n} \left(\int_M {}^t e_i(z; V_\rho \otimes \Delta) \overline{w(z)} e_j(z; V_\rho^\vee \otimes \Delta) \widehat{dz} \right) \equiv 0$$

for any fixed bases $\{e_i(z; V_\rho^{()} \otimes \Delta)\}$ of $H^0(M, V^{(*)} \otimes \Delta)$.*

This generalizes the standard theory of abelian theta functions and the Brill-Noether loci to non-abelian setting. For example, when $r = 1$, on the i -th Brill-Noether locus, the analogue of (iv) says that the analytic torsion may be calculated via the norm of the i -th partial derivatives of the standard theta functions. For details, see [F, Thm 4.9] and [ACGR].

Put all this together, we have then justified the convergence in the abelian case, namely, $r = 1$. As for general non-abelian cases, such a strong result has yet been obtained. Fortunately, what needed is a much weak result which we recall below:

Theorem 8. (Degenerations of Analytic Torisons [F, Thm 4.12]) *Let L be a line bundle of degree $d \geq g - 1$ such that $h^1(V_\rho \otimes L) = n > 0$ for a fixed $V_\rho \in \mathcal{M}_{M,r}(0)$. Then within a neighborhood U of V_ρ in $\mathcal{M}_{M,r}(0)$, the analytic torsions $T(V_{\rho(s)} \otimes L)$, which is positive whenever $h^1(V_{\rho(s)} \otimes L) = 0$, vanishes to order $2n$ at $V_\rho = V_{\rho(0)}$. In particular, near $s = 0$,*

$$T(V_{\rho(s)} \otimes L) = 4^n T(V_\rho \otimes L) \det[\overline{^t C(s)} C(s)] + O(\|s\|^{2n+1})$$

where for any orthonormal basis $\{e_i\}, \{e_j^*\}$ for $H^0(M, V_\rho \otimes L)$, $H^0(M, K_M \otimes (V_\rho \otimes L)^\vee)$ respectively:

$$C_{ij}(s) := \int_M {}^t e_i(z; V_\rho \otimes L) \overline{w} e_j^*(z; K_M \otimes (V_\rho \otimes L)^\vee) \widehat{dz}.$$

And for any x_1, \dots, x_{d+1-g} ,

$$\begin{aligned} & \det[\overline{^t C(s)} C(s)] \det[B(x_i, \overline{x_j}; V_{\rho(s)} \otimes L)] \\ &= \left| \det \begin{pmatrix} {}^t e_1(x_1 : V_\rho \otimes L) & \cdots & {}^t e_1(x_1 : V_\rho \otimes L) & C_{11}(s) & \cdots & C_{1n}(s) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ {}^t e_p(x_1 : V_\rho \otimes L) & \cdots & {}^t e_p(x_1 : V_\rho \otimes L) & C_{p1}(s) & \cdots & C_{pn}(s) \end{pmatrix} \right|^2 \\ &+ O(\|s\|^{2n+1}) \end{aligned}$$

as $\rho(s) \rightarrow \rho$ along any smooth curve transverse at ρ to the subvariety V_1 of all $V_\rho \in \mathcal{M}_{M,r}(0)$ with $h^1(M, V_\rho \otimes L) > 0$. Here $p = n + r(d + 1 - g)$.

Thus by the fact that the Bergman kernel admits logarithmic degeneration, we complete the proof of the convergence of the regularized integrations appeared in the proof of our zeta functions.

1.4.2 Asymptotics of Analytic Torsions

To establish the convergence of the infinite sum appeared in the definition of our zeta functions on m , we only need to understand

$$III(s) = \sum_{m > 2g-2} \int_{\mathcal{M}_{M,r}(rm)} \left(e^{\tau(M,V)} - 1 \right) (e^{-s})^{\chi(M,V)} d\mu,$$

by the discussion in §1.3. Note that for stable bundle of rank r and degree mr with $m > 2g - 2$, $h^0(M, V) = \chi(M, V) = r[m - (g - 1)]$ is a constant. So $T(V)$ is a constant function on $\mathcal{M}_{M,r}(mr)$. Thus using the natural isomorphism

$$\mathcal{M}_{M,r}(0) \simeq \mathcal{M}_{M,r}(mr), \quad V \mapsto V \otimes A^{\otimes m},$$

we have

$$III(s) = \sum_{m > g-1} \int_{V \in \mathcal{M}_{M,r}(0)} \left(e^{\tau(M, V \otimes A^{\otimes(m+g-1)})} - 1 \right) (e^{-s})^{rm} d\mu.$$

As such, then the convergence is guaranteed by the following results of Faltings, Miyaoka, Bismut-Vasserot:

Theorem 9. (See e.g., [BV, Thm 8]) As $m \rightarrow \infty$,

$$\frac{d}{ds} \zeta_\lambda^0(s, V \otimes A^{\otimes(m+1-1)}) \Big|_{\lambda=s=0} = O(m^r \log m).$$

Thus, in essence, we are dealing with the infinite summation

$$\sum_{m > g-1} \int_{M_{m,r}(0)} \left(e^{e^{-m^r \log m}} - 1 \right) (e^{-s})^{rm} d\mu$$

which is clearly convergent. This then completes the proof of Thm 1 stated in §1.2.

2 General Uniformity of Zetas

2.1 Number Fields: SL_n

2.1.1 Siegel's Volume Formula

For special linear group SL_n defined over \mathbb{Q} , there are 3 naturally associated groups, namely, the real Lie group $SL_n(\mathbb{R})$, its maximal compact subgroup $SO_n(\mathbb{R})$ and the full modular group $SL_n(\mathbb{Z})$. It is well known that the double quotient space $SL_n(\mathbb{Z}) \backslash SL_n(\mathbb{R}) / O_n(\mathbb{R})$ may be interpreted as the space of isometric classes of rank n lattices of volume one in the Euclidean space \mathbb{R}^n . Indeed, the metrics on \mathbb{R}^n are parametrized by matrices $A \cdot A^t$ with $A \in GL_n(\mathbb{R})$, and up to $O_n(\mathbb{R})$ -equivalence, the metric is uniquely determined by A . As such, then the lattice structures are finally determined modulo the automorphism group $SL_n(\mathbb{Z})$ of \mathbb{Z}^n .

Denote by $\mathbb{M}_{\mathbb{Q},n}[1]$ the moduli space of all full rank lattices in \mathbb{R}^n of volume one. The above discussion exposes the following

Fact 1. (Arithmetic versus Geometry) *There is a natural one-to-one correspondence*

$$SL_n(\mathbb{Z}) \backslash SL_n(\mathbb{R}) / SO_n(\mathbb{R}) \simeq \mathbb{M}_{\mathbb{Q},n}[1].$$

Associated to the natural measure on $SL_n(\mathbb{R})$, we may ask what is the corresponding volume of the above space. Surprisingly, while the space $SL_n(\mathbb{Z}) \backslash SL_n(\mathbb{R}) / SO_n(\mathbb{R})$, namely, $\mathbb{M}_{\mathbb{Q},n}[1]$, is highly non-abelian, or better, non-commutative, according to Siegel, its volume can be expressed in terms of the special values of Riemann zeta function, which is abelian in nature.

Fact 2. (Siegel) (Volume of Fundamental Domain)

$$m_{\mathbb{Q},n} := \text{Vol}(\mathbb{M}_{\mathbb{Q},n}[1]) = \widehat{\zeta}_{\mathbb{Q}}(1) \widehat{\zeta}_{\mathbb{Q}}(2) \cdots \widehat{\zeta}_{\mathbb{Q}}(n)$$

where $\widehat{\zeta}_{\mathbb{Q}}(s)$ denotes the complete Riemann zeta function and

$$\widehat{\zeta}_{\mathbb{Q}}(1) := \text{Res}_{s=1} \widehat{\zeta}_{\mathbb{Q}}(s).$$

2.1.2 Stability

Among all lattices, motivated by Mumford's fundamental work in algebraic geometry, we independently introduced the semi-stable lattices in our studies of non-abelian zeta functions. By definition, a lattice Λ is called *semi-stable* if for all sub-lattices Λ_1 of Λ

$$\text{Vol}(\Lambda_1)^{\text{rank} \Lambda} \geq \text{Vol}(\Lambda)^{\text{rank} \Lambda_1}.$$

Denote by $\mathbb{M}_{\mathbb{Q},n}^{\text{ss}}[1]$ the moduli space of rank n semi-stable lattices of volume 1. One checks that $\mathbb{M}_{\mathbb{Q},n}^{\text{ss}}[1]$ is a closed compact subset of $\mathbb{M}_{\mathbb{Q},n}[1]$. With the induced metric, define

$$m_{\mathbb{Q},n}^{\text{ss}} := \text{Vol}(\mathbb{M}_{\mathbb{Q},n}^{\text{ss}}[1]).$$

A natural question is what is the volume $u_{\mathbb{Q},n}$.

2.1.3 High rank non-abelian zeta functions

Similarly, denote by $\mathbb{M}_{\mathbb{Q},n}^{\text{ss}}$ the moduli space of rank n semi-stable lattices and by $\mathbb{M}_{\mathbb{Q},n}^{\text{ss}}[T]$ its volume T part. Then we have a natural decomposition

$$\mathbb{M}_{\mathbb{Q},n}^{\text{ss}} = \cup_{T \in \mathbb{R}_{>0}} \mathbb{M}_{\mathbb{Q},n}^{\text{ss}}[T].$$

Easily one checks that there is a natural isomorphism

$$\mathbb{M}_{\mathbb{Q},n}^{\text{ss}}[T] \simeq \mathbb{M}_{\mathbb{Q},n}^{\text{ss}}[T'] \quad \forall T, T' \in \mathbb{R}_{>0}. \quad (*)$$

Using the above measure on $\mathbb{M}_{\mathbb{Q},n}^{\text{ss}}[T]$ and the invariant Haar measure $\frac{dT}{T}$ on $\mathbb{R}_{>0}$, we obtain a natural measure $d\mu$ on $\mathbb{M}_{\mathbb{Q},n}^{\text{ss}}$.

Moreover, there is a genuine cohomology theory $h^i(F, \Lambda)$, $i = 0, 1$ for lattices Λ over number fields F for which the arithmetic analogue of the duality, the Riemann-Roch theorem, the vanishing theorem holds. For details, please refer to [W]. In the case of $F = \mathbb{Q}$,

$$h^0(\mathbb{Q}, \Lambda) = \log \left(\sum_{\mathbf{x} \in \Lambda} e^{-\pi \|\mathbf{x}\|^2} \right)$$

which was introduced earlier in [GS]. Denote by $d(\Lambda)$ the Arakelov degree of Λ , which over \mathbb{Q} is simply $-\log \text{Vol}(\Lambda)$. Following [W], define the associated *rank n non-abelian zeta function* $\hat{\zeta}_{\mathbb{Q},n}(s)$ by

$$\hat{\zeta}_{\mathbb{Q},n}(s) := \int_{\mathbb{M}_{\mathbb{Q},n}^{\text{ss}}} \left(e^{h^0(\mathbb{Q}, \Lambda)} - 1 \right) \cdot (e^{-s})^{d(\Lambda)} d\mu, \quad \text{Re}(s) > 1.$$

Then using the basic property of the above cohomology theory for h^i 's, namely the duality, the RR and the vanishing theorem, tautologically, we have the following

Fact 3. (Weng) (0) (**Relation with Abelian Zeta**)

$$\widehat{\zeta}_{\mathbb{Q},1}(s) = \widehat{\zeta}_{\mathbb{Q}}(s);$$

(i) (**Meromorphic Extension**) $\widehat{\zeta}_{\mathbb{Q},n}(s)$ is a well-defined holomorphic function in $\text{Re}(s) > 1$, and admits a unique meromorphic extension to the whole s -plane;

(ii) (**Functional Equation**)

$$\widehat{\zeta}_{\mathbb{Q},n}(1-s) = \widehat{\zeta}_{\mathbb{Q},n}(s);$$

(iii) (**Singularities**) $\widehat{\zeta}_{\mathbb{Q},n}(s)$ has only two singularities, all simple poles, at $s = 0, 1$. Moreover

$$\boxed{\text{Res}_{s=1} \widehat{\zeta}_{\mathbb{Q},n}(s) = m_{\mathbb{Q},n}^{\text{ss}} := \text{Vol}\left(\mathbb{M}_{\mathbb{Q},n}^{\text{ss}}[1]\right)}$$

In particular we see that $m_{\mathbb{Q},n}^{\text{ss}}$ is naturally related to the special value of the non-abelian zeta function $\widehat{\zeta}_{\mathbb{Q},n}(s)$.

2.1.4 Parabolic Reduction: Analytic Theory

The high rank zeta functions are closely related with Eisenstein series. In fact, we have

Fact 4. (Weng) (i) (**High Rank Zeta and Eisenstein Series**)

$$\widehat{\zeta}_{\mathbb{Q},n}(s) = \int_{\mathbb{M}_{\mathbb{Q},n}^{\text{ss}}[1]} \widehat{E}(\Lambda, s) d\mu = \int_{\left(SL_n(\mathbb{Z}) \backslash SL_n(\mathbb{R}) / O_n(\mathbb{R})\right)^{\text{ss}}} \widehat{E}^{SL_n/P_{n-1,1}}(\mathbf{1}, g; s).$$

Here $\widehat{E}(\Lambda, s)$ denotes the complete Eisenstein series associated to the lattice Λ , $\widehat{E}^{SL_n/P}(\mathbf{1}, g; *)$ denote the relative (complete) Eisenstein series on $SL_n(\mathbb{R})$ induced from the constant function $\mathbf{1}$ on the Levi factor of the maximal parabolic subgroup P and $P_{n-1,1}$ denotes the standard parabolic subgroup of SL_n corresponding to the partition $n = (n-1)+1$, and $\left(SL_n(\mathbb{Z}) \backslash SL_n(\mathbb{R}) / O_n(\mathbb{R})\right)^{\text{ss}}$ the part corresponding to the semi-stable lattices via Fact 1, which for our convenience will also be viewed as a subset of $SL_n(\mathbb{R})$;

(ii) (**Analytic Truncation versus Arithmetic Truncation**)

$$\Lambda^0 \mathbf{1} = \chi_{\left(SL_n(\mathbb{Z}) \backslash SL_n(\mathbb{R}) / O_n(\mathbb{R})\right)^{\text{ss}}}$$

Namely, Arthur's truncation of the constant function $\mathbf{1}$ is simply the characteristic function of the subset $\left(SL_n(\mathbb{Z}) \backslash SL_n(\mathbb{R}) / O_n(\mathbb{R})\right)^{\text{ss}}$ consisting of semi-stable points.

This is a number theoretic analogue of a result of Laffourge ([L]) on the relation between analytic truncation and arithmetic truncation for function fields.

Consequently,

$$\widehat{\zeta}_{\mathbb{Q},n}(s) = \int_{SL_n(\mathbb{Z}) \backslash SL_n(\mathbb{R}) / O_n(\mathbb{R})} \Lambda^0 \widehat{E}^{SL_n/P_{n-1,1}}(\mathbf{1}, g; s).$$

Here $\Lambda^0 \widehat{E}^{SL_n/P_{n-1,1}}(\mathbf{1}, g; s)$ denotes the Arthur's truncation of the Eisenstein series $\widehat{E}^{SL_n/P_{n-1,1}}(\mathbf{1}, g; s)$. On the other hand, by Langlands' theory of Eisenstein series, we know that

$$\widehat{E}^{SL_n/P_{n-1,1}}(\mathbf{1}, g; s) = \text{Res}_{\langle \lambda - \rho, \alpha_i^\vee \rangle = 0, i=1,2,\dots,n-2} \widehat{E}^{SL_n/P_{1,\dots,1}}(\mathbf{1}, g; \lambda)$$

where $\alpha_i = \alpha_i - \alpha_{i+1}$ denotes the simple roots of the root system A_{n-1} associated to SL_n , and $\rho = \frac{1}{2} \sum_{\alpha > 0} \alpha$ the Weyl vector.

With this, now notice that the moduli space $\mathbb{M}_{\mathbb{Q},n}[1]$ is compact, and that on the Levi of the Borel subgroup, $\mathbf{1}$ is cuspidal. So we can evaluate the *Eisenstein period*

$$\int_{SL_n(\mathbb{Z}) \backslash SL_n(\mathbb{R}) / O_n(\mathbb{R})} \Lambda^0 \widehat{E}^{SL_n/P_{1,\dots,1}}(\mathbf{1}, g; \lambda).$$

This then gives a very precise expression of non-abelian zeta function $\widehat{\zeta}_{\mathbb{Q},n}(s)$ as a combination of terms consisting of products of rational functions coming from the symmetry depending only on the root system, and abelian zeta functions. For details, please see [W]. As a direct consequence, we have the following

Fact 5. (*Weng*) (**Parabolic Reduction, Stability & the Volumes**)

$$m_{\mathbb{Q},n}^{\text{ss}} = \sum_{k \geq 1} (-1)^{k-1} \sum_{n_1 + \dots + n_k = n, n_i > 0} \frac{1}{\prod_{j=1}^{k-1} (n_j + n_{j+1})} \cdot \prod_{j=1}^k m_{\mathbb{Q},n_j}.$$

Geometrically, this means that the semi-stable part can be obtained from the fundamental domain associated to SL_n by deleting the tubular neighborhoods of cusps corresponding to parabolic subgroups which parametrize the same type of canonical flags of unstable lattices, and whose volumes, up to the lattice extensions, are completely determined by that associated to the simple factors of related Levi factors. Undoubtedly, this parabolic reduction is also the ‘heart’ of the theory of the truncations, both, analytic and arithmetic.

2.1.5 Parabolic Reduction: Geometric Theory

During our Sept, 2012's stay at IHES, Kontsevich introduced us their beautiful formula relating $m_{\mathbb{Q},n}$'s and $m_{\mathbb{Q},n}^{\text{ss}}$'s. This basic relation is obtained within their lecture notes on the wall-crossing ([KS]), with suitable renormalizations, their result can be stated as follows:

Fact 6. (*Kontsevich-Soibelman*) (**Parabolic Reduction, Stability & the Volume**)

$$\frac{1}{n} \cdot m_{\mathbb{Q},n} = \sum_{k \geq 1} \sum_{n_1 + \dots + n_k = n, n_i > 0} c_{n_1, n_2, \dots, n_k} \cdot \prod_{j=1}^k m_{\mathbb{Q}, n_j}^{\text{ss}}.$$

Here $c_{n_1, n_2, \dots, n_k} := \frac{1}{n_1(n_1 + n_2) \cdots (n_1 + n_2 + \dots + n_k) \cdots (n_{k-1} + n_k)n_k}$.

Indeed, the essence of this is the existence of the so-called canonical filtration, namely, the Harder-Narasimhan filtration of a lattice: *For a rank n lattice Λ , there exists a unique filtration of sub-lattices*

$$0 = \Lambda_0 \subset \Lambda_1 \subset \Lambda_2 \subset \dots \subset \Lambda_k = \Lambda$$

such that

- (i) $G_i(\Lambda) := \Lambda_i / \Lambda_{i+1}$ is semi-stable; and
- (ii) $\text{Vol}\left(G_i(\Lambda)\right)^{\text{rank}(G_{i+1}(\Lambda))} > \text{Vol}\left(G_{i+1}(\Lambda)\right)^{\text{rank}(G_i(\Lambda))}$.

2.2 Function Fields/ \mathbb{F}_q : SL_n

The interactions between studies of number fields and that of function fields (over finite fields) have been proven to be very fruitful, based on formal analogues between these two types of fields, despite the fact that many working mathematicians, not without their own reasons, believe otherwise. The works presented here are yet another group of beautiful examples.

2.2.1 Weil's Formula: Tamagawa Numbers

Motivated by Siegel's volume formula above, which originally was done in the theory of quadratic forms, Weil reinterpreted it in terms his famous Tamagawa number one conjecture ([Weil]). For SL_n , this goes as follows.

Let X be an irreducible, reduced, regular projective curve defined over V . Denotes its function field by F and its ring of adeles by \mathbb{A} . Fix a vector bundle E_0 of rank n on X with determinant λ .

Consider the group $SL_n(\mathbb{A})$ with $\mathbb{K}(E_0)$ the maximal compact subgroup associated to E . Then there is a natural morphism π from the quotient space $SL_n(F) \backslash SL_n(\mathbb{A}) / \mathbb{K}(E_0)$ to the stack $\mathbb{M}_{X,n}(\lambda)$ of rank n bundles with fixed determinant λ .

Fact 7. (*[HN], [DR]*) *The natural morphism*

$$\pi : SL_n(F) \backslash SL_n(\mathbb{A}) / \mathbb{K}(E_0) \rightarrow \mathbb{M}_{X,n}(\lambda)$$

is surjective with the fiber $\pi^{-1}(E_0^g)$ at the vector bundle E_0^g associated to $g \in SL_n(\mathbb{A})$ consisting of $\#(\mathbb{F}_q^ / \det \text{Aut}(E_0^g))$. Here $\det \text{Aut}(E_0^g)$ denotes the image of $\det \text{Aut}(E_0^g)$ in \mathbb{F}_q^* under the determinant mapping.*

Denote by $\mathbb{M}_{X,n}(d)$ the moduli stack of rank n bundle of degree d on X , and introduce the total mass for rank n and degree d bundles on X by

$$m_{X,n}(d) := \sum_{E \in \mathbb{M}_{X,n}(d)} \frac{1}{\# \text{Aut}(E)}.$$

Denote by

$$\widehat{\zeta}_X(s) := \zeta_X(s) \cdot (q^s)^{g-1}$$

the complete Artin zeta function associated to X , and

$$\widehat{\zeta}_X(1) := \text{Res}_{s=1} \widehat{\zeta}_X(s) \cdot \log q.$$

Then Weil's result that the Tamagawa number of the quotient space $SL_n(F) \backslash SL_n(\mathbb{A})$ equals one is equivalent to the following

Fact 8. (*Weil*) (**Tamagawa Number**)

$$m_{X,n}(d) = m_{X,n} := \widehat{\zeta}_X(1) \widehat{\zeta}_X(2) \cdots \widehat{\zeta}_X(n).$$

2.2.2 Non-Abelian Zeta Functions for X

Denote by $\mathbb{M}_{X,n}^{\text{ss}}(d)$ the moduli stack of rank n semi-stable bundle of degree d on X . Then define the *pure non-abelian zeta function of rank n for X* by

$$\widehat{\zeta}_{X,n}(s) := \sum_{k \in \mathbb{Z}} \sum_{E \in \mathbb{M}_{X,n}^{\text{ss}}(kd)} \frac{q^{h^0(X,E)} - 1}{\# \text{Aut}(E)} \cdot (q^{-s})^{\chi(X,E)}.$$

Write

$$\widehat{\zeta}_{X,n}(s) = \zeta_{X,n}(s) \cdot (q^s)^{n(g-1)}, \quad Z_{X,n}(t) := \zeta_{X,n}(s) \text{ with } t = q^{-s}$$

Introduce the partial mass of semi-stable bundles by

$$\alpha_{X,n}(d) := \sum_{E \in \mathbb{M}_{X,n}^{\text{ss}}(d)} \frac{q^{h^0(X,E)} - 1}{\# \text{Aut}(E)}, \quad \beta_{X,n}(d) := \sum_{E \in \mathbb{M}_{X,n}^{\text{ss}}(d)} \frac{1}{\# \text{Aut}(E)}.$$

Then tautologically,

$$Z_{X,n}(t) = \sum_{m=0}^{(g-1)-1} \alpha_{X,n}(mn) \cdot \left(T^m + Q^{(g-1)-m} \cdot T^{2(g-1)-m} \right) \\ + \alpha_{X,n}(n(g-1)) \cdot T^{g-1} + \frac{\beta_{X,r}(0)T^{2g-1}}{(1-QT)(1-T)} \cdot \left[(Q^g - 1) - (Q^g - Q)T \right].$$

where $T := t^n$ and $Q := q^n$. This exposes the following

Fact 9. (Weng) (i) **(Relation with Artin Zetas)** $\zeta_{X,1}(s) = \zeta_X(s)$, the Artin zeta function for X/\mathbb{F}_q ;
(ii) **(Rationality)** There exists a degree $2g$ polynomial $P_{X,r}(T) \in \mathbb{Q}[T]$ of T such that

$$Z_{X,r}(t) = \frac{P_{X,r}(T)}{(1-T)(1-QT)} \quad \text{with} \quad T = t^r, \quad Q = q^r;$$

(iii) **(Functional Equation)**

$$\widehat{\zeta}_{X,n}(1-s) = \widehat{\zeta}_{X,n}(s);$$

(iv) **(Residues)**

$$\widehat{\zeta}_{X,n}(1) := \text{Res}_{s=1} \widehat{\zeta}_{X,n}(s) \cdot \log Q = \beta_{X,n}(0) \left(= m_{X,n}^{\text{ss}}(0) \right).$$

2.2.3 Parabolic Reduction, Stability and the Mass: Geometric Theory

To go further, make a normalization by introducing

$$\widetilde{m}_{X,n}^{\text{ss}}(d) = \frac{1}{q^{\frac{n(n-1)}{2}(g-1)}} \cdot m_{X,n}^{\text{ss}}(d).$$

Then the parabolic reduction via the Harder-Narasimhan filtration leads to the following relation involving infinite sums:

Fact 10. ([HN], [DR]) **(Parabolic Reduction)**

$$m_{X,n}(d) = \sum_{k \geq 1} \sum_{n_1 + \dots + n_k = n, n_i > 0} \sum_{\substack{\frac{d_1}{n_1} > \dots > \frac{d_k}{n_k} \\ d_1 + \dots + d_k = d}} q^{-\sum_{i < j} (d_i n_j - d_j n_i)} \prod_{j=1}^k \widetilde{m}_{X,n_j}^{\text{ss}}(d_j).$$

2.2.4 Parabolic Reduction, Stability and the Mass: Combinatorial Aspect

With the above result, Zagier proved the following fundamental result, hidden in his paper on Verlinder formula:

Fact 11. (*[Z]*, see also *[WZ2]*) (**Parabolic Reduction, Stability & the Mass**)

$$\tilde{m}_{X,n}^{\text{ss}}(d) = \sum_{k \geq 1} (-1)^{k-1} \sum_{\substack{n_1 + \dots + n_k = n \\ n_i > 0, i=1, \dots, k}} \prod_{j=1}^{k-1} \frac{q^{(n_j + n_{j+1}) \cdot \{(n_1 + \dots + n_j) \cdot \frac{d}{n}\}}}{q^{(n_j + n_{j+1})} - 1} \cdot \prod_{j=1}^k m_{X,n_j}.$$

This formula should be compared with our Fact 5 for number fields. The structure are very much similar: in fact if we let $q \rightarrow 1$, then we would get the number theoretic identity there.

2.2.5 Parabolic Reduction, Stability and the Mass: New Formula

The above relation of Harder-Narasimhan, Ramanan-Desale and Zagier for function fields correspond to our own formula listed as Fact 5. So naturally, what should be the one appeared in the theory of parabolic reduction, stability and the volumes obtained by Kontsevich-Soibelman ([KS]).

Fact 12. (*[WZ2]*) For an ordered partition $n = n_1 + n_2 + \dots + n_k$, fix $\delta_i \in \{0, \dots, n_i - 1\}$, then fix $v_i \in [0, 1) \cap \mathbb{Q}$ satisfying

$$v_i \equiv \frac{\delta_i}{n_i} - \frac{\delta_{i+1}}{n_{i+1}} \pmod{1}.$$

Fix an n -th primitive root of unity ζ_n . Also set $N_i = n_1 + n_2 + \dots + n_i$ and $N'_i = n - N_i$ for $i = 1, 2, \dots, n$.

(i) (**On Average**)

$$n \cdot m_{X,n} = \sum_{k \geq 1} \sum_{\substack{n_1 + n_2 + \dots + n_k = n \\ n_i > 0, i=1, 2, \dots, k}} \sum_{\substack{\delta_i \in \{0, 1, \dots, n_i - 1\} \\ j=1, 2, \dots, k}} \prod_{i=1}^{k-1} \frac{q^{v_i N_i N'_i}}{q^{N_i N'_i} - 1} \cdot \prod_{j=1}^k \tilde{m}_{X,n_j}^{\text{ss}}(\delta_j).$$

(ii) (**Individuality**) For all $d = 0, 1, \dots, n - 1$,

$$m_{X,n} = \sum_{k \geq 1} \sum_{\substack{n_1 + n_2 + \dots + n_k = n \\ n_i > 0, i=1, 2, \dots, k}} \frac{1}{n} \\ \times \sum_{\substack{\delta_i \in \{0, 1, \dots, n_i - 1\} \\ j=1, 2, \dots, k}} \left(\sum_{\zeta_n: \zeta_n^n = 1} \zeta_n^{-d} \cdot \prod_{h=1}^{k-1} \frac{\zeta_n^{v_h N_h} q^{v_h N_h N'_h}}{\zeta_n^{N_h} q^{N_h N'_h} - 1} \right) \prod_{j=1}^k \tilde{m}_{X,n_j}^{\text{ss}}(\delta_j).$$

The first is obtained by taking average on d from the relation of Fact 11, while the second is obtained directly from that of Fact 11. With geometric picture in mind, formula (ii) should be further polished, so as to get everything done according to the real world structure. This may prove to be a bit complicated due to the fact that usually

$$\mathbb{M}_{X,n}(d) \not\cong \mathbb{M}_{X,n}(d'), \quad \forall d, d' \in \{0, 1, \dots, n-1\}, \quad d \neq d'.$$

This is very different from the cases for number fields, where we always have the isomorphism $(*)$ between different levels.

We remind the reader that while all relations in function field case are obtained using geometric methods, our basic relation for number fields are obtained analytically using Eisenstein series.

2.3 Parabolic Reduction, Stability and the Mass: General Reductive Groups

2.3.1 Parabolic Reduction Conjecture

Motivated by the above discussion, more generally, for a split reductive group G defined over a number field F , B a fixed Borel, ..., denote by $G(\mathbb{A})^{\text{ss}}$ the adelic elements of G corresponding to semi-stable principle G -lattices $([G])$. Write \mathbb{K}_G for the associated maximal compact subgroup. Also for a standard parabolic subgroup P , write its Levi decomposition as $P = UM$ with U the unipotent radical and M its Levi factor. Denote the corresponding simple decomposition of M as $\prod_i M_i$ with M_i 's the simple factors of M . Introduce invariants

$$m_{F,P} := \prod_i \text{Vol} \left(\mathbb{K}_{M_i} \backslash M_i^1(\mathbb{A}) / M_i(F) Z_{M_i^1(\mathbb{A})} \right)$$

and

$$m_{F,P}^{\text{ss}} := \prod_i \text{Vol} \left(\mathbb{K}_{M_i} \backslash M_i^1(\mathbb{A})^{\text{ss}} / M_i(F) Z_{M_i^1(\mathbb{A})} \right).$$

In parallel, we have similar constructions for function fields $F = \mathbb{F}_q(X)$.

Denote by

$$n_i := \#\{\alpha > 0 : \langle \rho, \alpha^\vee \rangle = i\} - \#\{\alpha > 0 : \langle \rho, \alpha^\vee \rangle = i+1\}$$

and by v_G the volume of $\{\sum_{\alpha \in \Delta} a_\alpha \alpha^\vee : a_\alpha \in [0, 1)\}$.

Fact 13. (*Langlands*) (**Volume of Fundamental Domain**) *For the field of rationals,*

$$\text{Vol} \left(\mathbb{K}_G \backslash G^1(\mathbb{A}) / G(\mathbb{Q}) Z_{G^1(\mathbb{A})} \right) = v_G \cdot \prod_{i \geq 1} \widehat{\zeta}(i)^{-n_i}.$$

Based on all this, then we have the following

Conjecture 10. (Parabolic Reduction) *Let G/F be a split reductive group with B/F a fixed Borel. Then, for each standard parabolic subgroup P of G , there exist constants $c_P \in \mathbb{Q}$, $e_P \in \mathbb{Q}_{>0}$, independent of F , such that*

$$m_{F;G} = \sum_P c_P \cdot m_{F;P}^{\text{ss}}, \quad m_{F;G}^{\text{ss}} = \sum_P \text{sgn}(P) \cdot e_P \cdot m_{F;P},$$

where P runs over all standard parabolic subgroups of G , and $\text{sgn}(P)$ denotes the sign of P .

Remark. Calculations in [Ad] for lower rank groups indicate that, for number fields, $\frac{1}{c_P} \in \mathbb{Z}_{>0}$. It would be very interesting to find a close formula for them.

2.3.2 Precise Formulation: Number fields

The exact values of e_P 's can be written out in terms of the associated root system. Indeed, if

$$W_0 := \left\{ w \in W : \{\alpha \in \Delta : w\alpha \in \Delta \cup \Phi^-\} = \Delta \right\},$$

then there is a natural one-to-one correspondence between W_0 and the set of subsets of Δ , and hence to the set of standard parabolic subgroups of G . Thus we will write

$$W_0 := \left\{ w_P : P \text{ standard parabolic subgroup} \right\},$$

and, for $w = w_P \in W_0$, write $J_P \subset \Delta$ the corresponding subset.

Conjecture 11. *Let G be a connected split reductive group with P its maximal parabolic subgroup defined over a number field F .*

(i) **(Relation to Zetas with Symmetries)**

$$m_{F;G}^{\text{ss}} = \text{Res}_{s=-c_P} \widehat{\zeta}_F^{(G,P)}(s) = \text{Res}_{\lambda=\rho} \omega_F^G(\lambda);$$

(ii) **(Parabolic Reduction, Stability & the Volumes)**

$$m_{F;G}^{\text{ss}} = \sum_P \frac{(-1)^{\text{rank}(P)}}{\prod_{\alpha \in \Delta \setminus w_P J_P} (1 - \langle w_P \rho, \alpha^\vee \rangle)} \cdot m_{F;P},$$

where P runs over all standard parabolic subgroups of G .

2.3.3 Precise Formulation: Function fields

In parallel, we have the following

Conjecture 12. *For a split reductive group G with P its maximal parabolic subgroup, both defined over the function field $\mathbb{F}_q(X)$ of an irreducible reduced regular projective curve X over \mathbb{F}_q ,*

(i) **(Relation to Zetas with Symmetries)**

$$\log q \cdot m_{F;G}^{\text{ss}}(0) = \text{Res}_{s=-c_P} \widehat{\zeta}_X^{(G,P)}(s) = \text{Res}_{\lambda=\rho} \omega_X^G(\lambda);$$

(ii) **(Parabolic Reduction, Stability & the Mass)**

$$m_{F;G}^{\text{ss}}(0) = \sum_P \frac{(-1)^{\text{rank}(P)}}{\prod_{\alpha \in \Delta \setminus w_P J_P} (q^{-\langle w_P \rho, \alpha^\vee \rangle + 1} - 1)} \cdot m_{F;P},$$

where P runs over all standard parabolic subgroups of G .

More generally, also to make this compatible with the paper of Laumon-Rapoport ([LR]), and for the purpose of understanding what happens for Riemann surfaces to be discussed below, next we consider arbitrary slope.

So in the follows we will freely use the notations of [LR]. Consider then the adelic space

$$\mathfrak{K}Z_{G(\mathbb{A})} \backslash G(\mathbb{A})^1 / G(K)$$

with \mathfrak{K} the maximal compact subgroup of $G(\mathbb{A})$, where \mathbb{A} denotes the adelic ring of K . For each $[\mathbf{g}]$ in this quotient space with $\mathbf{g} \in G(\mathbb{A})^1$, (up to universal constant factors independent of \mathbf{g} ,) define its mass by

$$m([\mathbf{g}]) := \frac{1}{|\mathbf{g}\mathfrak{K}\mathbf{g}^{-1} \cap G(K)|} = \frac{1}{|\text{Aut}(\mathcal{E}_{\mathbf{g}})|},$$

with $\mathcal{E}_{\mathbf{g}}$ the associated principal G -bundle, and the total mass by

$$m_{K;G;X} := \sum_{[\mathbf{g}]} m([\mathbf{g}])$$

where \mathbf{g} runs over all quotient classes. Similarly, for each $\nu'_G \in X_*(A'_G)$, set $m_{K;G;X}^{\text{ss}}(\nu'_G)$ be the partial (total) mass defined by running $[\mathbf{g}]$ over these whose associated G -bundles on X are semi-stable of slope ν'_G .

Conjecture 13. (i) **(Total Mass)**

$$m_{K;G}(\nu'_G) = m_{K;G} := \prod_{i \geq 1} \widehat{\zeta}_X(i)^{-n_i}.$$

(ii) **(Parabolic Reduction: Infinite Form)**

$$m_{K;G} = \sum_{P \in \mathcal{P}} \sum_{\substack{\nu'_P \in X_*(A'_P) \\ [\nu'_P]_G = \nu'_G}} \tau_P^G([\nu'_P]^G) q^{2m(P, \nu'_P)} m_{K;P}^{\text{ss}}(\nu'_P).$$

Here as above, $m_{K;P;X}^{\text{ss}}(\nu'_P) := \prod_{i=1}^k m_{K;M_{P,i}}^{\text{ss}}(\nu_{P,i}^{\dagger'})$ with $\nu_{P,i}^{\dagger'}$ the induced $M_{P,i}$ component of ν'_P .

(iii) **(Parabolic Reduction: Finite Form)**

$$m_{K;G}^{\text{ss}}(\nu'_G) = \sum_{P \in \mathcal{P}} (-1)^{\dim \mathfrak{a}_P^G} \sum_{\alpha \in \Delta_P} \frac{\prod_{\lambda \in \Lambda_P^G} q^{2\langle \rho_P, \alpha^\vee \rangle \langle \varpi_\alpha^G(\lambda) \rangle}}{q^{2\langle \rho_P, \alpha^\vee \rangle} - 1} \cdot m_{K;P}$$

where as above $m_{K;P} := \prod_{j=1}^k m_{K;M_{P,i}}$.

Clearly, (i) is the analogue of Siegel-Langlands' results on the volumes of fundamental domains associated to G over number fields, (ii) and (iii) coincides with Harder-Narasimhan ([HN]), Desale-Ramanan ([DR]), and Zagier ([Z], see also [WZ2]), respectively when $G = SL_r$. Following [LR], with Langlands' lemma, (iii) is a direct consequence of (i) and (ii). But (ii) is the argument of existence of Harder-Narasimhan filtration. So with ([RR]), what left is essentially (i).

As a direct consequence, then we can have Thms 3.2, 3.3 and 3.4 of [LR] on Poincare series for various moduli stacks of G -bundles, obtained by combining the method of Harder-Narasimhan ([HN]) and Atiyah-Bott ([AB]).

2.4 Riemann Surfaces

2.4.1 Zetas with Symmetries

By comparing the Parabolic Reduction for number fields and function fields, and note that the natural ∞ -factor corresponding to $\frac{1}{1-q^{-s}}$ is $\Gamma_{\mathbb{R}}(s) := \pi^{-\frac{s}{2}} \Gamma(\frac{s}{2})$, we propose the following definition of zetas for Riemann surfaces with symmetries.

Set $\widehat{\zeta}_M(s) := \widehat{\zeta}_{M,1}(s)$ and call it the zeta function of M . Then for any pair (G, P) with G a reductive group and P its maximal parabolic subgroup, all defined over $\mathbb{C}(M)$, the period of G over M is defined by

$$\omega_X^G(\lambda) := \sum_{w \in W} (-1)^{|G|} \prod_{\alpha \in \Delta} \Gamma_{\mathbb{R}}(-\langle w\lambda - \rho, \alpha^\vee \rangle) \cdot \prod_{\alpha > 0, w\alpha < 0} \frac{\widehat{\zeta}_X(\langle \lambda, \alpha^\vee \rangle)}{\widehat{\zeta}_X(\langle \lambda, \alpha^\vee \rangle + 1)},$$

and the period of (G, P) over M as

$$\omega_M^{(G,P)}(\lambda_p) := \text{Res}_{\substack{\langle \lambda - \rho, \beta_i^\vee \rangle = 0 \\ \{\beta_1, \dots, \beta_{|P|}\} \subset \Delta \leftrightarrow P}} \left(\omega_X^G(\lambda) \right).$$

This then finally leads to the zeta functions with symmetries associated to (G, P) for M as

$$\widehat{\zeta}_M^{G/P}(s) := \text{Norm} \left(\omega_M^{(G,P)}(\lambda_p) \right)$$

where Norm stands the normalization obtained similarly as for number fields and/or function fields.

2.4.2 General Uniformity of Zetas: Special One

Motivated by our works on number fields and function fields, we formulate the following:

Conjecture 14. (Special Uniformity) *There exist constants c_n, a_n, b_n such that*

$$\widehat{\zeta}_{M,r}(s) = c_n \cdot \widehat{\zeta}_M^{SL(n)/P_{n-1,1}}(a_n s + b_n).$$

2.4.3 Two Dimensional Gauge Theory

For the volume of the moduli spaces $\mathcal{M}_{M,r}(0)$, Witten in his two dimensional gauge theory established the following

Theorem 15. (Volume Formula, See e.g. [Wi])

$$\text{Vol}(\mathcal{M}_{X,n}(0)) = n \cdot \left(\frac{\text{Vol}(SU(n))}{(2\pi)^{n^2-1}} \right)^{2g-2} \sum_{\rho} \frac{1}{\dim \rho^{2g-2}}$$

where ρ runs over all irreducible representations of $SU(n)$.

In particular, we note that these volumes are independent of the Riemann surfaces. Such an independence is in fact embedded in the main result of conformal field theory claiming that conformal blocks, namely, the global sections of multiples of our determinant line bundle λ above, are essentially independent of the complex structures used: after all, they are projectively flat. Thus, particularly, their dimensions, and hence its leading term, namely our volume up to normalization, according to the Hirzebruch-Riemann-Roch, are independent of the complex structures used.

2.4.4 General Uniformity of Zetas: Tamagawa Number Conjecture

- $F = \mathbb{C}(X)$: function field of X
- \mathbb{A} : ring of adeles of F
- E_0 : fixed bundle E_0 w/ determinant λ , say $E = \bigoplus_{i=0}^{n-1} \lambda$
- $\mathbb{K} = \prod_{p \in X} \mathbb{K}_p$ w/ $\mathbb{K}_p := SL_n(\mathbb{C}[[z_p]])$

Fact 14. (Adelic Interpretation of Space of Bundles) *The space of rank n bundles may be identified with*

$$\left(\prod_p GL_n(\mathbb{C}[[z_p]]) \right) \backslash GL_n(\mathbb{A}) / GL_n(F)$$

Moreover, we have the following

Conjecture 16. (Tamagawa Numbers) *There exist natural measures such that*

$$(i) \tau\left(SL_n(F) \backslash SL_n(\mathbb{A})\right) = 1$$

$$(ii) m(\mathbb{K}) = \widehat{\zeta}_M(1) \widehat{\zeta}_M(2) \cdots \widehat{\zeta}_M(n).$$

2.4.5 General Uniformity of Zetas: Parabolic Reduction

Here the difficulty is to count $\text{Aut}(\mathcal{E})$ for a vector bundle \mathcal{E} on M : even for stable bundles, it gives $\mathbb{C}^* = GL_1(\mathbb{C})$. Any natural count for $GL_n(\mathbb{C})$, or more generally, the space \mathbb{C}^n and the torus $(\mathbb{C}^*)^m$. to make the discussion go forward, assume that we do have a natural way to count all of them (in a compatible way), then we would get a nice definition of the mass for each bundle \mathbb{E} and hence also $m(SL_n; M)$, the corresponding $m^{\text{ss}}(SL_n; M)$, and moreover $m(SL_n; P)$, $m^{\text{ss}}(SL_n; P)$ for each parabolic subgroup P . By a parallel way of thinking, we then would arrive as the following

Conjecture 17. (General Uniformity of Zetas: Riemann Surfaces) *For a connected compact Riemann surface M ,*

(i) *(Total Mass)*

$$m_{M;G}(\nu'_G) = m_{M;G} := \prod_{i \geq 1} \widehat{\zeta}_M(i)^{-n_i}.$$

(ii) **(Parabolic Reduction: Finite Form)**

$$m_{M;G}^{\text{ss}}(\nu'_G) = \sum_{P \in \mathcal{P}} (-1)^{\dim \mathfrak{a}_P^G} \sum_{\alpha \in \Delta_P} \Gamma_{\mathbb{R}}(\langle \rho_P, \alpha^\vee \rangle - 1) \cdot m_{M;P}.$$

2.4.6 A Rational Function

Another way to approach is to use h^0 directly to create non-abelian zeta functions for Riemann surfaces. This goes as follows:

Let X be a compact Riemann surface of genus g . Denote by $\mathcal{M}_{X,n}^{\text{ss}}(d)$ the moduli space of semi-stable bundles of rank n and degree d on X and $W_{X,n}^{\geq i} := \{V \in \mathcal{M}_{X,n}^{\text{ss}}(d) : h^0(X, V) \geq i\}$ the Brill-Noether filtration with $W_{X,n}^=i := W_{X,n}^{\geq i} \setminus W_{X,n}^{\geq i+1}$. By the above discussion, we get $d\mu := d\mu_{X,n}(d)$ naturally induced volume forms on $W_{X,n}^=i(d)$.

Fix $\pi \in \mathbb{R}$, introduce the *regularized integral*

$$\begin{aligned} & \int_{\mathcal{M}_{X,n}(d)}^{\#} \left(\pi^{h^0(X,V)} - 1 \right) \left(\pi^{-s} \right)^{\chi(X,V)} d\mu_{X,n}(d) \\ &:= \sum_{i \geq 0} \int_{W_{X,n}^=i(d)} \pi^{h^0(X,V)} \left(\pi^{-s} \right)^{\chi(X,V)} d\mu_{X,n}(d) - \int_{\mathcal{M}_{X,n}(d)} \left(\pi^{-s} \right)^{\chi(X,V)} d\mu_{X,n}(d) \\ &= \sum_{i \geq 0} \pi^{i-sd+ns(g-1)} \text{Vol}\left(W_{X,n}^=i(d)\right) - \pi^{-sd+n(g-1)s} \text{Vol}\left(\mathcal{M}_{X,n}(d)\right). \end{aligned}$$

Then a simple version of rank n zeta function of X is defined by

$$\tilde{\zeta}_{X,n}(s) := \sum_{\alpha \geq 0} \int_{\mathcal{M}_{X,n}(\alpha \cdot n)}^{\#} \left(\pi^{h^0(X,V)} - 1 \right) \cdot \left(\pi^{-s} \right)^{\chi(X,V)} \cdot d\mu.$$

Tautologically, using the Riemann-Roch, the duality and the vanishing for semi-stable bundles, we have the following

Theorem 18. (Zeta Facts) (i) **(Rationality)** $\tilde{\zeta}_{X,n}(s)$ is rational in $T = t^n = \pi^{-ns}$ with $t = \pi^{-s}$
(ii) **(Functional Equation)** $\tilde{\zeta}_{X,n}(1-s) = \tilde{\zeta}_{X,n}(s)$
(iii) **(Singularities)** $\tilde{\zeta}_{X,n}(s)$ admits only two singularities, all simple poles, at $s = 0, 1$

$$\text{Res}_{s=1} \tilde{\zeta}_{X,n}(s) = \text{Vol}(\mathcal{M}_{X,n}(0)).$$

Note that by Witten's volume formula and our parabolic reduction conjecture, if this were the right zeta, the special values of this rational function would be independent of X . Consequently, the functions themselves would all be the same.

Similar statement can be applied to $\hat{\zeta}_{M,n}(s)$. For this we have to use a result of Ramanujan-Harder and Deninger ([D]), which claims that certain analytic functions are uniquely determined by their special values at almost all positive integers. This then suggests that our high rank zetas are independent of M , an assertion which should be understood in the framework of the existence of projective flat connections in the theory of conformal blocks.

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